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Implementing a Pattern and Structure Mathematics Awareness Program (PASMAP) in Kindergarten

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This paper provides an interim report of a large empirical evaluation study in progress. An intervention was implemented to evaluate the effectiveness of the Pattern and Structure Mathematical Awareness Program (PASMAP) on Kindergarten students' mathematical development. Four large schools (two from Sydney and two from Brisbane), 16 teachers and their 316 students participated in the first phase of a 2-year longitudinal study. Eight of 16 classes implemented the PASMAP program over three school terms. This paper provides an overview of key aspects of the intervention, and preliminary analysis of the impact of PASMAP on students' representation, abstraction and generalisation of mathematical ideas.

Virtually all mathematics is based on pattern and structure. By mathematical *pattern*, we mean any predictable regularity involving number, space or measure. Examples are friezes, number sequences, units of measure and geometrical figures. By *structure*, we mean the way in which the various elements are organised and related. Thus, a frieze might be constructed by iterating a single "unit of repeat"; the structure of a number sequence may be expressed in an algebraic formula; and the structure of a geometrical figure is shown by its various properties. Structural thinking can emerge from, or underlie mathematical concepts, procedures and relationships. Mason, Stephens and Watson (2009) view structural thinking as more than simply recognising elements or properties of a relationship but having a deeper awareness of how those properties are used, explicated or connected.

Early childhood research on pattern and structure

There is an increasing body of research into young children's structural development of mathematics and early algebraic reasoning. Recent research in the area of number (Hunting, 2003; Mulligan & Vergnaud, 2006; Thomas, Mulligan & Goldin, 2002; van Nes & de Lange, 2007; Ellemor Collins & Wright, 2009; Young-Loveridge, 2002), patterning and reasoning (Clements & Sarama, 2009; English, 2004; English & Watters, 2005; Papic, 2007), spatial measurement (Outhred & Mitchelmore, 2000; Slovin & Dougherty, 2004), early algebra (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; Warren & Cooper, 2008), and data modelling (Lehrer, 2007) have all shown how progress in student's mathematical understanding depends on a grasp of underlying structure.

A suite of studies by Mulligan and her colleagues (Mulligan, 2009) have suggested that children who have developed an awareness of structure in one aspect of the early mathematics learning also tend to show a structural awareness in other aspects. Mulligan and Mitchelmore (2009) postulated the existence of a general construct called Awareness of Mathematical Pattern and Structure (AMPS). To test this hypothesis, they developed a Pattern and Structure Assessment (PASA), consisting of items from across the Year 1 curriculum, and categorised responses both as correct/incorrect and as showing one of four levels of structural development. They found not only that each student tended to show a single structural level in all their responses, but also that this level was strongly correlated with the total number of correct responses. They therefore argued that AMPS could be measured using the PASA interview, and that AMPS was indeed associated with mathematical understanding.

The questions naturally arise, is it possible to improve students' AMPS by an appropriate intervention, and if so, does their general mathematical achievement also improve? A number of studies supported this conjecture. For example, in collaboration with the teachers at one preschool, Papic (2007) developed a 6-month intervention focussed on repeating and spatial and patterns. Not only did the children outperform a comparison group on a patterning assessment instrument administered at the end of the intervention, but nine months later about half of them were able to continue growing patterns (a task that none of the comparison children could do). Furthermore, the intervention children outperformed the comparison children on a state-wide test of early numeracy. Papic then revised and implemented the intervention program with several groups of preschoolers and cooperating professionals in contrasting settings. She found that preschoolers who are provided with opportunities to engage in mathematical experiences that promote emergent generalisation are capable of abstracting complex patterns before they start formal schooling (Papic, Mulligan, & Mitchelmore, 2009). The crucial aspect is establishment of a unit of repeat, exposure to a variety of patterns in differing modes and orientations, and scaffolding by an adult to justify and transfer these patterns to other forms. These experiences led to the development of informal mathematical inscriptions and number knowledge.

Mulligan and colleagues developed a Pattern and Structure Mathematics Awareness Program (PASMAP) that focuses explicitly on raising primary school students' awareness of mathematical pattern and structure via a variety of well-connected pattern-eliciting experiences. Studies have included an extensive, whole-school professional development exercise across Kindergarten to Year 6; two year-long, single teacher studies in Years 1 and 2; and an intensive, 15-week individualised program with a small group of low-ability Kindergarten children (For details, see Mulligan, 2009). Many individual cases have been documented showing astonishing changes in children's structural awareness and development of mathematical concepts well beyond that expected for their age level. Some evidence has emerged that PASMAP also has an effect on their scores on independent mathematics assessments. More importantly the PASAMP aims to promote simple or 'emergent generalisation' in young children's mathematical thinking across a range of concepts.

The studies cited above lend strong support to the hypothesis that teaching young children about pattern and structure should lead to a general improvement in the quality of their mathematical understanding. However, none of the studies had a sufficiently large or representative sample, most lacked a comparison group and there was insufficient opportunity to track and describe in depth, the growth of structural development. The current study was therefore designed to evaluate the effects of PASMAP on student mathematical development in the first year of formal schooling.

Method

Participants: A purposive sample of four large primary schools, two in Sydney and two in Brisbane, representing 316 students from a diverse range of socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. Two different mathematics programs were implemented: in each school, two Kindergarten teachers implemented the PASMAP and two implemented their standard program. The PASMAP framework was embedded into the standard Kindergarten mathematics curriculum, enabling schools to meet the required system-based learning outcomes for New South Wales and Queensland, respectively.

Professional development: A designated researcher/teacher visited each teacher on a weekly basis and equivalent professional development for both pairs of teachers was provided by the research team. A one-day professional development program was provided at the initial stage of the project independently for each teacher group (standard and PASMAP program). The framework was outlined for independent use by the PASMAP teachers with an accompanying sequence of learning experiences described in terms of syllabus outcomes and core components of the PASMAP. There was sufficient scope in the program for teachers to develop their own teaching/learning sequences that differentiated for individuals. Incremental features of the program were introduced by the research team gradually, at approximately the same pace and with equivalent mentoring for each teacher, over three school terms (May to December). The implementation time varied considerably between classes and schools, ranging from one 50-minute lesson per week to 5 one-hour lessons per week. Incidental mathematical experiences incorporating features of PASMAP supplemented regular classroom lessons, such as puzzles that were completed at home or other activities that incorporated patterning such as music and visual arts.

Assessment: All students were pre- and post- tested with the standardised *I Can Do Maths* (ICDM) (Doig & de Lemos, 2000); from the pre-test data two 'focus' groups of five children in each class were selected from the upper and lower quartiles, respectively. These 160 students were interviewed using a new version of a 20-item *Pattern and Structure Assessment* (PASA). They were monitored closely by the teacher and the research assistant by collecting detailed observation notes, digital recordings of their mathematics learning and work samples, and other classroom-based assessment data. These data formed the basis of digital profiles for each student. At the end of the year, students were re-tested with ICDM and the PASA was administered to the majority of the 'focus' students. Follow-up assessments will be carried out in September 2010. The ICDM and the PASA data was supported by a larger and richer data set providing direct evidence of the students' daily classroom-based mathematics learning. These data included daily assessment tasks accompanying each element/lesson which was completed by individual students. The same process was replicated for 'focus' students in the regular program. Other classroom-based (teacher assessment), school-based (whole cohort testing) and system-based (e.g., Schedule for early Number Assessment (SENA) in NSW) provided supplementary evidence of the 'focus' students' mathematical progress.

Analysis: The classroom-based qualitative data of student learning, including video data, is currently being analysed to describe students' structural development of mathematics (AMPS). Microgenetic analyses for each PASMAP 'focus' student aims to build a profile by (i) describing the 'tracked' developmental pathway/s of their mathematical concepts and processes, (ii) the quality of the underlying structural characteristics, (iii) evidence of salient features or relationships built by the student between components or concepts, and (iv) evidence of emergent generalisations and reasoning to support these. To complement the primary analyses, ICDM and the PASA scores will be compared for PASMAP and non-PASMAP students. Further analyses of the students' responses on selected PASA items will support the qualitative analyses of features of structural development using procedures trialled in previous studies (Mulligan & Mitchelmore, 2009). Evaluation data also includes teachers' views of the impact of the program on student learning (teacher interviews), their perceptions of themselves as professionals and the impact of the study on school-wide professional development and change.

The Pattern and Structure Mathematics Awareness Program Intervention

The program is innovative in its conceptual framework and the way learning experiences are scaffolded, where children are encouraged to seek out and represent pattern and structure across different concepts and transfer this awareness other concepts. In other words, the aim is to promote generalisation in early mathematical thinking. It focuses on fundamental processes such as simple and complex repetitions, growing patterns and functions, unitising and multiplicative structure also common to units of measure; spatial structuring, the spatial properties of congruence and similarity, and transformation. Table 1 describes some key elements of the first phase of the program.

Table 1: *Key components of the PASMAP intervention*

Component	Focus
Counting with shapes, staircase patterns Rhythmic and perceptual counting	Counting by twos, threes, fours, fives using regular shapes Constructing simple patterns using perceptual counting.
Repetition Simple AB and complex patterns AAB (with and without models)	Constructing, drawing, symbolising and justifying linear and cyclic patterns using a variety of materials.
Unit of repeat	Chunking, ordering, symbolising and translating.
Similarity and congruence (2D shapes)	Comparing and drawing similar triangles and squares, distinguishing congruence.
Benchmarking	Constructing and partitioning length; assigning symbols to equal sized units
Symmetry and transformations	Identifying symmetry through matching and congruence.
Subitising	Identifying number and shape in subitising patterns, three to nine. Spatial structuring of subitising patterns.
Grids	Identifying number of units in simple grids, 2 x 2, 3 x 3, 4 x 4, 5 x 5 squares and 2 x 3 rectangles. Deconstructing and reconstructing from memory the spatial properties of grids.

Arrays	Identifying number of units in simple arrays, 1 x 2, 1 x 3, 2 x 2, 3 x 3. Deconstructing and reconstructing from memory the spatial properties of arrays.
Data representation: functional thinking	Constructing tables of data, representing simple counting patterns as a function

Emphasis is also laid on developing number concepts through pattern and structure such as an emphasis on counting patterns and their relation to measurement, geometry and data exploration, and the structure of mathematical number operations such as equivalence, commutativity and inverse operations. The development of visual memory is critical to promoting abstraction and symbolisation. In this paper, we present examples of structural development drawn from classroom-based data of children working with the PASMAP program. To illustrate the differential effect of the program, we have selected examples from the same PASMAP class in Sydney, but from different ends of the ability spectrum. Qualitative analyses of digital recordings and students' representations provided complementary evidence of their invented symbolisations and generalisations in repetitions and growing patterns. Improvements in mathematical processes such as skip counting, multiplicative thinking, unitising and partitioning, similarity and congruence, and area measurement were observed. For example, we tracked the development of individuals' imagistic representations for explicit features of structural development such as unitising, congruence and co-linearity.

Figures 1, 2 and 3 show a 5-year old's progress in identifying spatial characteristics of a 2 x 2 square grid and representing this over time. The gain in structural understanding is obvious. Evidence that the student has conceptualised the properties of a square and the structure of the 2 x 2 grid is provided from the following excerpt of transcript accompanying the representation in Figure 3: " I made them the same…the squares have the same on each side . It doesnt matter if they are big or small, they got the same sides. You have to put only the squares that you need. They have to be same size… I know they have to match if they are on top and on top you know… I made a four with two and two ".

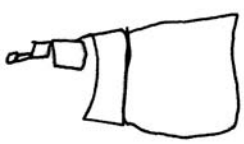	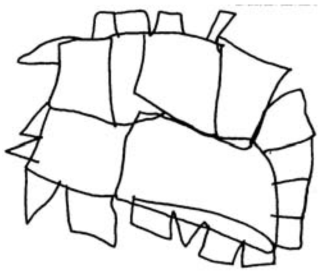	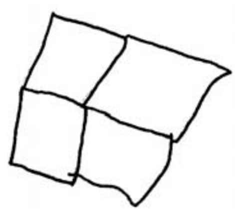
Figure 1. Drawing of 2 x 2 squares, initial attempt.	Figure 2. Drawing of 2 x 2 squares, later attempt.	Figure 3. Drawing of 2 x 2 squares, final attempt.

Children were engaged in a series of daily tasks filling empty regular ten frame cards to show one to one matching, spatial and counting patterns, groups, quantities, and addition combinations 1 to 10. Ten frames with regular dot patterns were also used to support learning. As well the children experimented with the spatial structure of the frame. As an assessment task children were required to draw the frame from memory and describe how they did this and why the frame was used. Figures 4, 5 and 6 compare different attempts to provide the correct ten frame structure, 2 x 5 units.

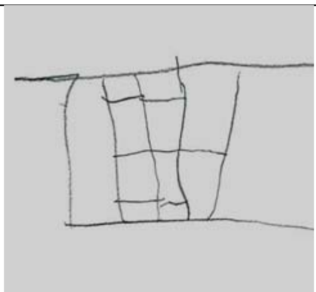		
Figure 4. Drawing of ten frame from memory, initial attempt.	Figure 5. Drawing of ten frame, later attempt.	Figure 6. Drawing of ten frame, final attempt.

In another task the children had to recall their use of pattern cards depicting the pattern of squares i.e., 1, 2 x 2, 3 x 3, 4 x 4, 5x5 square grid cards. This pattern was linked to prior use of simple grid patterns as depicted in Figure 3 and the counting patterns of multiples. Again they were required to recall from memory the pattern in order and represent it accurately. Figures 7 and 8 show two widely different responses.

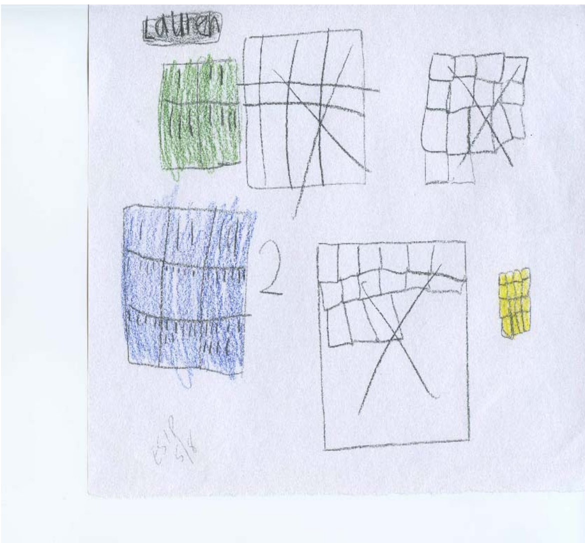	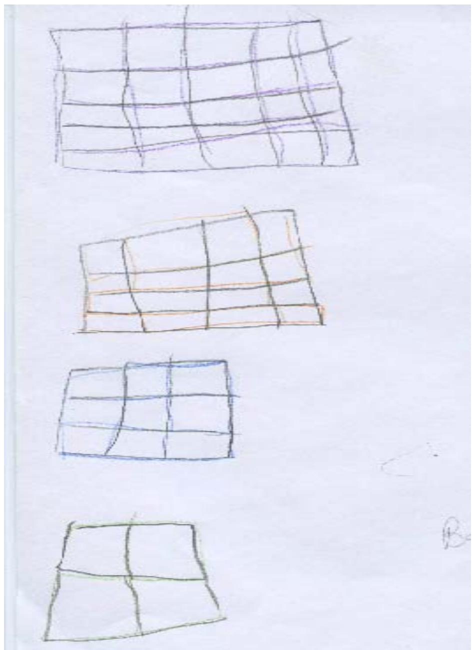
Figure 7. Drawing of pattern of squares from memory, initial attempt.	Figure 8. Drawing of pattern of squares, later attempt.

The child was able to visualise the structure of the pattern accurately because they focused on both the shape and the increasing row and column structure more so than remembering a numerical pattern.

Discussion

Both groups of students showed impressive growth in essential mathematics learning outcomes as described by state syllabus and measured by the ICDM test. There were no significant differences found between PASMAP and regular students on pre- and post-tests scores on this standardised measure. However, the qualitative data, tracking of the ‘focus’ students indicates at this stage of the analysis stark differences in the way they developed their mathematical knowledge and reasoning skills. Only the children in the PASMAP program made direct connections between mathematical ideas and processes and formed emergent generalisations. Some of the more able students used one aspect of pattern and structure to build new and more complex concepts. Gradually these connections became more like systems of learning that had common structural features. Goldin refers to these as autonomous powerful systems that become independent over time (Thomas et al., 2002). It would be expected that a focus on pattern, structure, representation and justification advantaged the PASMAP students. It is also possible that students in the regular program made similar connections between mathematical ideas and used them effectively but they were not given opportunities to describe or explain their reasoning or develop emergent generalised thinking. All young children might be given appropriate opportunities to develop skills in reasoning, problem-solving, justification and argumentation (Perry & Dockett, 2008).

Implications for Curriculum and Pedagogy

There has been increasing interest in using a structural approach, especially as it relates to algebraic understanding, in mathematics curricula throughout Australia and internationally. A structural approach focuses on patterns and relationships that lead to abstract ideas and generalisations. In the forthcoming Australian National Curriculum (ACARA, 2010), Number and Algebra strands are aligned with Problem Solving and Reasoning Proficiencies. "An algebraic perspective can enrich the teaching of number…and the integration of number and algebra, especially representations of relationships can give more meaning to the study of algebra in the secondary years. This combination incorporates pattern and/or structure and includes functions, sets and logic". Further, the integration of measurement and geometry, and statistics and probability brings new opportunities to develop a structural approach. However, structural development has not previously been central to mathematics syllabi or early years' learning. Perhaps current curriculum structure of parallel strands has dissuaded teachers and students from making important connections and encourages the teaching of discrete concepts and procedures. Further there are few links to other curriculum areas such as Science and Technology where common aspects such as measurement and data exploration are underpinned by similar structural features. The proposed PASMAP will enable professionals to develop and evaluate a new approach with flexibility–one that integrates patterns and structural relationships in mathematics across concepts so that a more holistic outcome is achieved.

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